Method of Two-dimensional Nonlinear Laplace Transformation for Solving the Navier – Stokes Equation

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ABSTRACT: ***This paper introduces a new nonlinear Laplace transforms allowing to reduce the Navier – Stokes equation to ordinary Riccati's one. It is mathematically proved that the no-linear summand $v\frac{dv}{dz}$ in the Navier – Stokes equation can be reduced to that expressing the multiplication of the operators $f_1^{\circ}(\xi,t)\cdot f_2^{\circ}(\xi,t)$, where the functions $f_1^{\circ}(\xi,t)$ and $f_2^{\circ}(\xi,t)$ are images of the functions v and $\frac{dv}{dz}$, correspondingly. Such an approach gives the opportunity to get analytical solution of the problem of viscous fluid motion in a pipe. The algorithm may be used to solve the majority of algebraic nonlinear problems of mathematical physics.***

Keywords: Navier-Stokes equation, liquid motion, nonlinear Laplace transform, multiplication of operators, images, Riccati equation

I. INTRODUCTION

The vast majority of mechanics and physics phenomena are described by the nonlinear partial differential equations that, generally speaking, are analytically integrated in exceptional cases. Commonly used methods for solving them are known numerical ones whose accuracy depends on the number and quality of used approximations (e.g. the difference method considered by Goldberg [1]). Besides, it is worthwhile to mark some analytical techniques that might be applied for solving partial nonlinear differential equations. For example, a new numerical-analytical technique, which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the nonlinear root-mean-square (rms) envelope dynamics developed by Fedorova et al [2-5]. Such an approach may be useful in all the models in which it is possible and reasonable to reduce all complicated problems relating to statistical distributions to the problems described by systems of nonlinear ordinary/partial differential equations. On the other hand, the good review of symmetry technique is presented in the "Symmetries of Equations of Quantum Mechanics" by Fushchich and Nikitin [6]. The book deals with the analysis of old (classical) and new (non-Lie) symmetries of the fundamental equations of quantum mechanics and classical field theory, and with classification and algebraic-theoretical deduction of equations of motion of arbitrary spin particles in both Poincaré invariant approach ([7-16] and references in [6]). Perhaps, a number of these or other analytical methods (see conclusion of the paper) are acceptable to solve, under some simplifying conditions, fluid mechanics problems, and the Navier-Stokes equation in particular. They, however, cannot be used as universal methods for solving all the known algebraic nonlinear partial differential equations including the Navier-Stokes' in the general case. The task we set was just to develop a universal technique providing analytically correct solution in all cases; and the nonlinear Laplace transform is what we propose to use for this purpose.

It's accepted that the operational methods (in particular, the Laplace transform) could not be used for solving nonlinear equations. This opinion is based on the fact, that this transformation could not linearize the nonlinear summand introduced into the Navier - Stokes equation. Nevertheless, in the present paper we consider a new integral transformation on the base of Laplace transform, which gives an opportunity to integrate analytically and correctly the above-mentioned equation. This transformation is "more nonlinear" than the ordinary Laplace transform and it is the property that may lead to accurate solutions of the problems of fluids motion.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

For simplicity, we are going to investigate z-problem of fluid motion, i.e. non - stationary problem of one-dimensional longitudinal (along the axis z) fluid motion in cylindrical pipe with velocity uniformly averaged by cross-section. Our purpose is to find the distribution function of velocity v (z, t). Under such a statement to find the required function v (z, t) it is necessary to solve the following equation

$$\rho\frac{\partial v}{\partial t}+\rho v\frac{\partial v}{\partial z}=\eta\frac{\partial^2 v}{\partial z^2}-\frac{\partial P}{\partial z}, \qquad (1)$$

where the function of pressure P = P (z, t) is taken as previously known. The Eqn. (1) should be solved under the next initial and boundary conditions

$$v(z,t=0)=0\,,\; v(z=0,t)=V_0(1-e^{-\chi t})\,,$$
$$\frac{d\,v(z=0,t)}{d\,z}=\phi \qquad (2)$$

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Existing integral transformations are considered as not sufficient to integrate the Eqn. (1) because of the impossibility to use them to acquire the image of nonlinear summand $v\frac{dv}{dz}$ in the left side of the equation. This difficulty may be overcome by using so-called nonlinear Laplace transform by coordinate z as follows:

$$\int_0^\infty\int_0^\infty f(z,t)e^{-2\xi z}(dz)^2 \qquad (3)$$

The coordinate Laplace transform recently has successfully been used to solve some differential equations [17,18]. The new aspect in the transform (3) is the double integrating with the kernel equal to *exp (-2ξ z)*. Such an approach allows us to use the theorem of operators multiplication taking place in the operational calculus. Indeed, if the integrals

$$f_1^\circ(\xi,t)=\int_0^\infty v(z,t)\,e^{-\xi z}dz \qquad (4)$$

and

$$f_2^\circ(\xi,t)=\int_0^\infty \frac{\partial v(z,t)}{\partial z}\,e^{-\xi z}dz \qquad (5)$$

are absolutely convergent at $\mathrm{Re}(\xi)\succ\sigma_a$, then the double integral

$$f_1^\circ(\xi,t)\cdot f_2^\circ(\xi,t)=\int_0^\infty\int_0^\infty v(z,t)\frac{\partial v(z,t)}{\partial z}e^{-2\xi z}(dz)^2 \qquad (6)$$

will absolutely converge as well (see the conclusions of Ditkin and Prudnikov [19]).

The integral (6) is the nonlinear transformation like (3) from the function $v\frac{dv}{dz}$. Therefore, when using the transformation (3) for nonlinear summand in the Navier - Stokes Eqn. (1), this one - according to the basic laws of the operational calculus - will be expressed in images through multiplication of the operators $f_1^\circ(\xi,t)$ and $f_2^\circ(\xi,t)$, where the operators $f_1^\circ(\xi,t)$ and $f_2^\circ(\xi,t)$ are defined by the formulas (4) and (5) respectively.

III. CONVOLUTION OF OPERATORS OF THE FUNCTIONS V AND $\frac{dv}{dz}$

It follows from validity of the integral transformation (3) and, consequently, the relationship (6) that the convolution of the functions v and $\frac{dv}{dz}$ should exist, thereto, the image of this convolution has to be equal to $f_1^\circ(\xi,t)\cdot f_2^\circ(\xi,t)$.

Theorem. If the integrals (4) and (5) are absolutely convergent, then the function $V^\circ(\xi,t)=f_1^\circ(\xi,t)\cdot f_2^\circ(\xi,t)$ is the ordinary Laplace transform of the function

$$V(z,t)=\int_0^z v(z-\vartheta)\frac{\partial v(\vartheta)}{\partial\vartheta}d\vartheta \qquad (7)$$

Proof. Since the integrals (4) and (5) are absolutely convergent, it can be concluded that the double integral

$$f_1^\circ(\xi,t)\cdot f_2^\circ(\xi,t)=\int_0^\infty\int_0^\infty v(y)\frac{\partial v(w)}{\partial w}e^{-\xi(y+w)}dy\,dw$$

is also absolutely convergent. If this integral is subjected to variables substitution y + w = z, y = ϑ, then the integration area comes into the plane part restricted by the straight lines $\vartheta=0$ and ϑ = z, as it has been established by Ditkin and Prudnikov [19]. Now, using Fubini theorem, it may be concluded that the integral (7) is valid for all values of z and moreover

$$V^\circ(\xi,t)=f_1^\circ(\xi,t)\cdot f_2^\circ(\xi,t)=$$
$$=\int_0^\infty e^{-\xi z}dz\int_0^z v(z-\vartheta)\frac{\partial v(\vartheta)}{\partial\vartheta}d\vartheta$$

The last relationship proves the correctness of the expression (6).

The function *V (z,t)* is the convolution of the two original functions v and $\frac{dv}{dz}$, and therefore, the property of commutability for this function should be valid. In other words, the equality

$$\int_0^z v(z-\vartheta)\frac{\partial v(\vartheta)}{\partial\vartheta}d\vartheta=\int_0^z v(\vartheta)\frac{\partial v(z-\vartheta)}{\partial\vartheta}d\vartheta \qquad (8)$$

should take place. Correctness of the formula (8) can be proved quite easily. Let's calculate the left integral in the relationship (8):

$$\int_0^z v(z-\vartheta)\frac{\partial v(\vartheta)}{\partial\vartheta}d\vartheta=v(z-\vartheta)\times v(\vartheta)\Big|_0^z+$$
$$+\int_0^z v(\vartheta)\frac{\partial v(z-\vartheta)}{\partial\vartheta}d\vartheta$$

The obvious equality to zero of the first summand in the right side just leads to the required equation. Now, it is necessary to show that

$$V^\circ(\xi,t)=\int_0^\infty V(z,t)e^{-\xi z}dz\,,$$

but this aspect follows directly from the Borel theorem.

Hence, while using the integral transformation (3) for the Navier-Stokes equation, the nonlinear summand in images may be represented as a multiplication of the two operators $v_1^\circ(\xi,t)$ and $v_2^\circ(\xi,t)$. Each of them is defined above in order that the final equation relative to the function $v^\circ(\xi,t)$ in the images also to be nonlinear. However, since this equation is an ordinary nonlinear one, it can also be analytically, correctly resolved.

IV. SOLVING THE PROBLEM OF NON-STATIONARY MOTION OF VISCOUS FLUID

Let's now solve the Navier-Stokes equation directly. The transformation (3) can be used to solve the Eqn. (1). Keeping in mind the boundary conditions (2), as well as that of permanence of the pressure P (z,t) on the boundary z=0 during the whole process of fluid motion, we obtains the following equation in complex plane *(ξ ,t)*:

$$\rho\frac{1}{\xi}\frac{dv^\circ}{dt}+\rho v^\circ\left[\xi v^\circ-V_0(1-e^{-\chi t})\right]= \\ =\frac{\eta}{\xi}\left[\phi+\xi^2 v^\circ-\xi V_0(1-e^{-\chi t})\right]-\frac{1}{\xi}(\xi P^\circ-P_0) \quad , \quad (9)$$

where $v^\circ=v^\circ(\xi,t)=\int_0^\infty v(z,t)e^{-\xi z}dz$, but the function P° - the image of the pressure in the complex plane *(ξ,t)*. After certain calculations the equation (9) may be reduced to

$$\frac{dv^\circ}{dt}+a_1(v^\circ)^2-a_2 v^\circ=a_3 \quad (10)$$

which is the general Riccati's equation; hereafter the coefficients a_1, a_2 and a_3 (generally speaking, they are functions of time) are determined as

$$a_1=\xi^2,\ a_2=\frac{\eta}{\rho}\xi^2+\xi V_0(1-e^{-\chi t}),$$
$$a_3=\frac{1}{\rho}\left\{(P_0-\xi P^\circ)-\eta\xi V_0(1-e^{-\chi t})+\eta\phi\right\}$$

As is well known, the general Riccati's equation is tightly connected with the linear differential equations of the second order. If for the considered time interval $0<t<\infty$, the functions a_1 and a_2 are continuous and the a_1 is differentiated (it is not difficult to show that these conditions are naturally valid), then each solution to the $v^\circ(\xi,t)$ of the Riccati's equation by means of the transformation

$$v^\circ(\xi,t)=\frac{U'(\xi,t)}{\xi^2 U(\xi,t)} \quad (11)$$

may be reduced to non-zero solution of the linear differential equation of the second order:

$$U''-a_2U'-a_1a_3U=0 \quad (12)$$

The transformation (11) is essential for the linear Eqn. (12) to be often resolved much easier than the original Riccati's equation.

The Eqn. (12) with the time depended coefficients a_1 and a_2 is the most general one for determining the function $U(\xi,t)$ and consequently the velocity v (z,t). Let's consider some partial cases, for example, one will investigate the area close to the beginning of pipe; in images this condition corresponds to approximation $\xi\to\infty$. Taking into consideration this approximation, after appropriate estimation of the parameters a_1, a_2 and a_3 the Eqn. (12) will be reduced to

$$U''-\frac{\eta}{\rho}\xi^2U'-\xi^2(\alpha_1+\alpha_2 e^{-\chi t})=0, \quad (13)$$

where the constants α_1 and α_2 are found as follows:

$$\alpha_1=\frac{1}{\rho}\left\{(P_0-\xi P^\circ)-\eta\xi V_0\right\},\ \alpha_2=\frac{\eta}{\rho}\xi V_0$$

Solution to the Eqn. (13) should be searched for as was noted by Kamke [20]

$$U(\xi,t)=\exp\left[\frac{\eta}{2\rho}\xi^2 t\right]\aleph_\nu(\theta);$$
$$\theta=\frac{2\xi}{\chi}\sqrt{\alpha_2}\,e^{-\chi t/2},\ \nu=\frac{\xi}{\chi}\left(\frac{\eta^2}{\rho^2}\xi^2-4\alpha_1\right)^{1/2}, \quad (14)$$

To find the required solution, one should calculate the derivatives $J_\nu'(\theta)$ and $Y_\nu'(\theta)$. Let us use properties of the Bessel functions in differentiating [21]

$$\frac{1}{x}\frac{d}{dx}\left[x^\nu\aleph_\nu(x)\right]=x^{\nu-1}\aleph_{\nu-1}(x) \quad (15)$$

The computations made lead to

$$\frac{d}{dx}\aleph_\nu(x)=\aleph_{\nu-1}(x)-\frac{\nu}{x}\aleph_\nu(x) \quad (16)$$

Then, for derivative of the function $\frac{d\aleph_\nu(\theta)}{dt}$ the next expression is obtained

$$\frac{d\aleph_\nu(\theta)}{dt}=\frac{\chi}{2}\nu\aleph_\nu(\theta)-e^{-\chi t/2}\xi\sqrt{\alpha_2}\,\aleph_{\nu-1}(\theta)$$

According to the last formula, the velocity $v^\circ(\xi,t)$ in images may be written as follows:

$$v^\circ(\xi,t)=C+\frac{\eta}{2\rho}\left(1+\left[1+\frac{4\rho}{\eta\xi}P^\circ\right]^{1/2}\right)- \\ -\frac{\sqrt{\alpha_2}}{\xi}\frac{\aleph_{\nu-1}(\theta)}{\aleph_\nu(\theta)}e^{-\chi t/2} \quad , \quad (17)$$

herein the constant C is introduced into the relationship (17) for justifying the initial condition (2). In the considered case, the functions $\aleph_\nu(\theta)$ and $\aleph_{\nu-1}(\theta)$ may be simplified in accordance with the properties of Bessel functions given by Olver in [21]:

$$\aleph_\nu=\frac{C_1}{(2\pi\nu)^{1/2}}\left(\frac{e\theta}{2\nu}\right)^\nu-C_2\left(\frac{2}{\pi\nu}\right)^{1/2}\left(\frac{e\theta}{2\nu}\right)^{-\nu};$$
$$\aleph_{\nu-1}=\frac{C_1}{\left(2\pi(\nu-1)\right)^{1/2}}\left(\frac{e\theta}{2(\nu-1)}\right)^{\nu-1} \\ -C_2\left(\frac{2}{\pi(\nu-1)}\right)^{1/2}\left(\frac{e\theta}{2(\nu-1)}\right)^{-(\nu-1)} \quad (18)$$

After this representation for the ratio $\frac{\aleph_{\nu-1}}{\aleph_\nu}$, one gets the following expression:

$$\frac{\aleph_{\nu-1}}{\aleph_\nu}=\left(\frac{\nu-1}{\nu}\right)^{1/2}\frac{(\nu-1)^{\nu-1}}{\nu^\nu}\frac{e\theta}{2}\approx \\ \approx\left(\frac{\nu-1}{\nu}\right)^{1/2}\frac{1}{\nu}\frac{e\theta}{2}=e\left(\frac{\rho V_0}{\eta\xi\left[1+\frac{4\rho}{\eta^2\xi}P^\circ\right]}\right)^{1/2}e^{-\chi t/2} \quad (19)$$

In deducing the last formula we neglect the second summands in the expansions (18), because under great values of ν these summands do not effect in fact on the final result as well as the condition $\nu\approx\nu-1$ is used which is valid at considered values of the order of the Bessel functions.

The relationship (19) may be simplified if to keep in mind the approximation $\xi\to\infty$. Then, expanding the denominator in the formula (19) and having restricted ourselves by two terms we get

$$\frac{\aleph_{\nu-1}}{\aleph_\nu} \approx \frac{e}{2}\left(\frac{\rho}{\eta\xi}V_0\right)^{1/2} e^{-\chi t/2}\left(1-\frac{2\rho}{\eta^2\xi}P^\circ\right) \quad (20)$$

The formula (20) allows to write the expression for fluid velocity in the images $v^\circ(\xi,t)$:

$$v^\circ(\xi,t) = C + \frac{\eta}{2\rho}\left(1+\left[1+\frac{4\rho}{\eta\xi}P^\circ\right]^{1/2}\right) - \frac{eV_0}{2\xi}\left(1-\frac{2\rho}{\eta^2\xi}P^\circ\right)e^{-\chi t} \quad (21)$$

The constant *C* can be found from the initial condition (2). After realizing appropriate calculations for fluid velocity in images we may finally obtain

$$v^\circ(\xi,t) = \frac{eV_0}{2\xi}\left(1-\frac{2\rho}{\eta^2\xi}P^\circ\right)(1-e^{-\chi t}) \quad (22)$$

For finding the obvious form of the function $v^\circ(\xi,t)$ it is necessary to know the distribution P (z). For the problem let's assume that the pressure (which is constant by time) changes by coordinate z by exponential law

$$P(z) = P_0 e^{-\kappa z}, \quad (23)$$

In our opinion, the law (23) is the most general one out of possible ways describing the character of fluid pressure reducing along the pipe length. In images, the law (23) will correspond to $P^\circ = \frac{P_0}{\kappa+\xi}$. The final expression is

$$v^\circ(\xi,t) = \frac{eV_0}{2\xi}\left(1-\frac{2\rho}{\eta^2}\frac{P_0}{\xi(\kappa+\xi)}\right)(1-e^{-\chi t}) \quad (24)$$

Returning to real plane (z,t), we obtain the following formula for the velocity:

$$v(z,t) = \frac{eV_0}{2}\left(1-\frac{2\rho}{\eta^2}P_0\frac{e^{-\kappa z}+\kappa z-1}{\kappa^2}\right)\times (1-e^{-\chi t}) \quad (25)$$

Taking into account that the values of magnitudes z and κ are small, it may be transformed to

$$v(z,t) = \frac{eV_0}{2}\left(1-\frac{\rho}{\eta^2}P_0 z^2\right)(1-e^{-\chi t}) \quad (26)$$

The relationship (26) bears witness about square law of fluid velocity change (decreasing) along the pipe axis near the pipe beginning. By analyzing the expression (26), it can conclude that under the problem's conditions the fluid velocity begins to arise from zero till certain stationary value $v_\infty(z)$ which is defined by

$$v_\infty(z) = \frac{eV_0}{2}\left(1-\frac{\rho}{\eta^2}P_0 z^2\right)$$

The velocity value (26) is correct only if the two first summands in the above expansion for square root $\left(1+\frac{4\rho}{\eta^2\xi}P^\circ\right)^{-1/2}$ (valid for heavy viscous fluid and small pressures) are taken into account. In account of the summands of higher degrees, there will be different formulas for the fluid velocity. For example, taking into account the third summand in the expansion

$$\left(1+\frac{4\rho}{\eta^2\xi}P^\circ\right)^{-1/2} \approx 1-\frac{2\rho}{\eta^2\xi}P^\circ + \frac{6\rho^2}{\eta^4\xi^2}(P^\circ)^2$$

one gets additional term for the (25), which describes increment of the liquid velocity by the axis z. As a result, for the function v (z,t) there is the sign-alternating series

$$v(z,t) = \frac{eV_0}{2}\left(1-\frac{\rho}{\eta^2}P_0 z^2 + \frac{3\rho^2}{4\eta^4}P_0^2 z^4\right)\times (1-e^{-\chi t}) \quad (27)$$

The relationship (27) is obtained under the same conditions as the formulas (25) and (26).

The analysis of (27) shows that the value of the liquid velocity undergoes a minimum. It can be proved that this extremum is placed on a distance z_{cr} from the forward end of the pipe, where

$$z_{cr} = \eta\sqrt{\frac{2}{3P_0\rho}}$$

In other words, despite the exponentially decreasing pressure P by coordinate z, the liquid velocity has a minimum. Probably, this minimum may be explained by relaxation properties of the liquid. Thereto, as it follows from our calculations, under certain conditions, the number of extremums may increase considering the summands of higher order in the above expansion[1)]. Positions of these extremums are determined from the algebraic equation

$$\sum_{j=0}^{N}(-1)^j \alpha_j z^{2j} = 0, \quad (28)$$

where $\alpha_0 = 1$; sign before the coefficient α_0 depends upon whether the number *N* is odd or even. Thus, one may conclude that in the areas close to the forward end of the pipe, the liquid velocity is a complex function like damping vibrations. The formula (27) allows to find an asymptotic value of the liquid velocity at steady regime of the motion:

$$v(z,t) = \frac{eV_0}{2}\left(1-\frac{\rho}{\eta^2}P_0 z^2 + \frac{3\rho^2}{4\eta^4}P_0^2 z^4\right) \quad (29)$$

Validity of the final formulas (26) and (29) actually depends on how correct the expansions (18) of the Bessel functions $\aleph_\nu(\theta)$ and $\aleph_{\nu-1}(\theta)$ are. Hence, before comparing the results obtained in this paper with other available approximations, it is necessary to verify precision of the expansions listed by Olver [21]. Other cases of practical interest (distance remote from the forward end of the pipe, motion under the laser

[1)] The fig.1 clearly shows the difference between formulas (26) and (29): taking of the second-degree summand in the expansion into account leads to appearance of the extremum. It is reasonable that the liquid will behave as described by the formulas (26) and (29) only at values of z exceeding z_c calculated by (28) and the previous expression. At the interval z < z_{cr}, however, the liquid velocity will monotonically reduce.

radiation action) have been considered by the author [22].

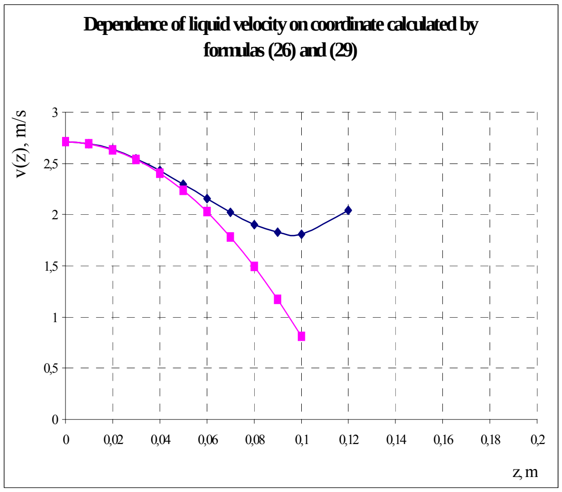

Fig. 1 Dependences of liquid velocity on coordinate calculated by formulas (26) and (29): Red color corresponds to formula (26), blue color - (29). Calculations have been carried out for some typical values of commercial liquid parameters: ρ = 700 kg/m^3, η=1000 kg/m·sec. Checking them is explained by conditions of the problem.

V. CONCLUSION

The method of nonlinear two-dimensional Laplace transforms developed in the present paper is principally a new technique for integrating nonlinear partial differential equations. It is worthwhile to note that precision of the result obtained depends upon accuracy of expansions for the Bessel functions $\aleph_\nu(\theta)$ and $\aleph_{\nu-1}(\theta)$ like formulas (18). As it can be shown, the Navier - Stokes equation can be, in the most general cases, reduced to the equation of Riccati with variable coefficients. Hence, after realizing the nonlinear transformation (3) the Navier - Stokes equation can be reduced only to solving the Riccati's equation of certain difficulty. In principle, this conclusion is not speculative since time-independent solution of the Navier-Stokes equation (steady case) may be exactly taken from the Riccati's equation [23]. The theory of the last equation is developed quite thoroughly, so the above-mentioned transformation gives an opportunity to get analytically correct solution to the great number of problems in hydrodynamics, heat physics, mechanics etc. for various processes having practical interest.

In this regard, the method developed in the paper distinguishes from various recent methods, e.g. those developed by Rogerson and Yeow [24] realized by means of artificially introduced functions (a typical example, the Papkovich–Fadle eigenfunctions recently considered in [24]).and further manipulations with them. Thereto, for each problem of liquid motion one should look for appropriate Papkovich–Fadle eigen-functions. In comparison with the latter the method of nonlinear Laplace transforms is much more universal and gives an opportunity without great efforts to get imaging equation relative to the function $v^\circ(\xi, t)$. After finding the $v^\circ(\xi, t)$ one can return to real plane (z, t) and define original function $v(z, t)$.

Finally, we would like to touch upon some generalization of the two-dimensional Laplace transforms. We consider the using of the above-developed technique for solving nonlinear partial differential equations with three and higher order. To do so, it is necessary to apply the nonlinear Laplace transform of appropriate order. For example, if one has third-order partial nonlinear differential equation, then for solving it one can use the Laplace transform of the form

$$\int_0^\infty\int_0^\infty\int_0^\infty f(z,t)e^{-3\xi z}(dz)^3 \qquad (30)$$

This procedure will lead to third-order ordinary differential equation relative to the image $f^\circ(\xi, t)$ which then can be solved without any problem.

VI. NOMENCLATURE AND GREEK SYMBOLS

a_1, a_2 and a_3 – some coefficients including into the equation of Riccati (10),

N – number of summands in the expansion for $\left(1+\frac{4\rho}{\eta^2\xi}P^\circ\right)^{-1/2}$,

V_0 - stationary value of fluid velocity at point z = 0, at $t\to\infty$,

v (z, t)- axial velocity of liquid in cylindrical pipe,

$v^\circ(\xi, t)$ - images of the liquid velocity,

$U(\xi, t)$-additional function for finding the liquid velocity,

α_j – constant coefficients in the expansion (28),

ρ - the density of moving fluid,

η - its viscosity,

ξ - the Laplace coordinate parameter,

χ and ϕ - certain constant parameters characterizing relaxation properties of the fluid and its velocity gradient in the pipe beginning, respectively,

σ_a - the absolute abscissa of convergence,

κ - certain constant coefficient,

$\aleph_\nu(\theta)$ -the linear combination of the Bessel functions $J_\nu(\theta)$ and $Y_\nu(\theta)$ like $\aleph_\nu(\theta)=C_1J_\nu(\theta)+C_2Y_\nu(\theta)$,

ν - order of the Bessel functions; due to the conditions on involved problem runs to infinity,

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